GORENSTEIN WEAK DIMENSION OF A COHERENT POWER SERIES RINGS

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Abstract. We compute the Gorenstein weak dimension of a coherent power series rings over a commutative rings and we show that, in general, $\mathrm{Gwdim}\,(R) \leq 1$ does not imply that R is an arithmetical ring.

1. Introduction

Throughout this paper, all rings are commutative with identity element, and all modules are unital.

Let R be a ring, and let M be an R-module. As usual we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension and flat dimension of M. By $\mathrm{gldim}\,(R)$ and $\mathrm{wdim}\,(R)$ we denote, respectively, the classical global dimension and weak dimension of R.

For a two-sided Noetherian ring R, Auslander and Bridger [1] introduced the G-dimension, $\mathrm{Gdim}_R(M)$, for every finitely generated R-module M. They showed that there is an inequality $\mathrm{Gdim}_R(M) \leq \mathrm{pd}_R(M)$ for all finite R-modules M, and equality holds if $\mathrm{pd}_R(M)$ is finite.

Several decades later, Enochs and Jenda [9, 10] defined the notion of Gorenstein projective dimension (G-projective dimension for short), as an extension of G-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (G-injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda and Torrecillas [11] introduced the Gorenstein flat dimension. Some references are [3, 6, 7, 10, 11, 14].

Recently in [4], the authors started the study of the notions global Gorenstein dimensions of ring R, which are denoted by $\mathrm{GPD}\,(R)$, $\mathrm{GID}\,(R)$, and $\mathrm{Gwdim}\,(R)$ and defined as follows:

(1) $\mathrm{GPD}\,(R) = \sup\{\mathrm{Gpd}_R(M) \mid M \text{ be an } R\text{-module}\}$

(2) $\mathrm{GID}\,(R) = \sup\{\mathrm{Gid}_R(M) \mid M \text{ be an } R\text{-module}\}$

(3) $\mathrm{Gwdim}\,(R) = \sup\{\mathrm{Gfd}_R(M) \mid M \text{ be an } R\text{-module}\}$

They proved that, for any ring R, $\mathrm{Gwdim}\,(R) \leq \mathrm{GID}\,(R) = \mathrm{GPD}\,(R)$ ([4, Theorems 2.1 and 2.11]). So, according to the terminology of the classical theory of homological dimensions of rings, the common value of $\mathrm{GPD}\,(R)$ and $\mathrm{GID}\,(R)$ is called Gorenstein global dimension of R, and denoted by $\mathrm{Ggldim}\,(R)$. They also proved that the Gorenstein global and weak dimensions are refinement of the classical global and weak dimensions of rings. That means $\mathrm{Ggldim}\,(R) \leq \mathrm{gldim}\,(R)$ (resp.

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Gwdim $(R) \leq$ wdim (R)), and equality holds if wdim (R)) is finite ([4, Propositions 2.12]).

In [15] Jøndrup and Small gave a connection between a weak dimension of a coherent power series ring over a commutative ring R and the weak dimension of R, see also [13, Theorem 8.1.1]. In the following we recall this result:

Theorem 1.1. *Let R be a ring, and let x be an indeterminate over R. If $R[[x]]$ is a coherent ring, then $wdim\,(R[[x]]) = wdim\,(R) + 1$.*

In this paper, we give an extension of Theorem 1.1 to the Gorenstein weak dimension.

We know that if wdim $(R) \leq 1$, then R is an arithmetical ring (see for instance [2]). Now it is natural to ask the following question: "Does Gwdim $(R) \leq 1$ imply that R is an arithmetical ring?" In Theorem 2.12, we give a negative answer to this question. More precisely, we prove: Let $(R, \mathfrak{m})$ be a local quasi-Frobenius ring which is not a field. Then Gwdim $(R[[X]]) = 1$ but $R[[X]]$ is not an arithmetical ring.

2. Gorenstein weak dimension

First we recall the notion of strongly Gorenstein projective module which is introduced in [3].

Definition 2.1. A module M is said to be strongly Gorenstein projective (*SG*-projective for short), if there exists an exact sequence of the form:

$$\mathbf{P} = \ \cdots \rightarrow P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \rightarrow \cdots$$

where P is a projective R-module and f is an endomorphism of P, such that $M \cong \mathrm{Im}\,(f)$ and such that $\mathbf{Hom}(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective module.

These strongly Gorenstein projective modules has a simple characterization, and they are used to characterize the Gorenstein projective modules. We recall the following two results which are [3, Propositions 2.9] and [3, Theorem 2.7]:

Proposition 2.2. *A module M is strongly Gorenstein projective if, and only if, there exists a short exact sequence of modules:*

$$0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$$

where P is projective and $Ext\,(M, Q) = 0$ for any projective module Q.

Theorem 2.3. *A module is Gorenstein projective if, and only if, it is a direct summand of a strongly Gorenstein projective module.*

Lemma 2.4. *Let R be a ring and let X be an indeterminate over R and M an $R[[X]]$-module. Then X is a nonzero divisor on M if and only if $Tor_{R[[X]]}(M, R) = 0$.*

Proof. Let $\varphi_X : M \longrightarrow M$ be the homomorphism of $R[[X]]$-module such that $\varphi_X(m) = Xm$ for every $m \in M$. Consider the short exact sequence of $R[[X]]$-modules

$$(\star) \quad 0 \longrightarrow R[[X]] \xrightarrow{\mu_X} R[[X]] \longrightarrow R \cong R[[X]]/XR[[X]] \longrightarrow 0$$

where μ_X is the multiplication by X. The following sequence is induced from $(\star)$

$$0 \longrightarrow \mathrm{Tor}_{R[[X]]}(M,R) \longrightarrow R[[X]]\otimes_{R[[X]]}M \stackrel{1_M\otimes\mu_X}{\longrightarrow} R[[X]]\otimes_{R[[X]]}M \longrightarrow R\otimes_{R[[X]]}M \longrightarrow 0$$

By [17, Theorem 8.13] the $R[[X]]$-morphism $1_M\otimes\mu_X$ is multiplication by X. So, in the following diagram all squares are commutative

$$\begin{array}{ccccccccc}
0\to & \mathrm{Tor}_{R[[X]]}(M,R) & \to & R[[X]]\otimes_{R[[X]]}M & \stackrel{1_M\otimes\mu_X}{\to} & R[[X]]\otimes_{R[[X]]}M & \to & R\otimes_{R[[X]]}M\to 0 \\
 & & & \wr\,\| & & \wr\,\| & & \wr\,\| \\
0\to & Ker(\varphi_X) & \to & M & \stackrel{\varphi_X}{\to} & M & \to & M/XM \quad\to 0
\end{array}$$

Therefore, $\mathrm{Ker}\,(\varphi_X)\cong \mathrm{Tor}_{R[[X]]}(M,R)$ and hence X is a nonzero divisor on M if, and only if, $\mathrm{Tor}_{R[[X]]}(M,R)=0$. □

Lemma 2.5. *Let R be a ring and X an indeterminate over R such that $R[[X]]$ is coherent. If M is a finitely presented $R[[X]]$-module such that X is a nonzero divisor on M then $Gpd_{R[[X]]}(M)\leq Gpd_R(M/XM)$.*

Proof. First note that R is a coherent ring by [13, Theorem 4.1.1(1)]. In addition, X is contained in the Jacobson radical of $R[[X]]$. Let M be a finitely presented $R[[X]]$-module M over which X is a nonzero divisor and put $n=\mathrm{Gpd}_R(M/XM)$. We may assume that n is finite.
The proof will be by induction on n.
If M/XM is a Gorenstein projective R-module, then by using [8, Proposition 10.2.6 (1)⇔(10)], the proof is the same as the one of [6, Corollary 1.4.6] (note that X is an element of the Jacobson radical of $R[[X]]$ and so we may use the Nakayama's Lemma in the proof of [6, Corollary 1.4.6]. In the original proof we use the *Local* condition) .
Now, assume that $n>0$ and consider the short exact sequence of $R[[X]]$-modules $0\longrightarrow G\longrightarrow P\longrightarrow M\longrightarrow 0$ where P is a finitely presented projective $R[[X]]$-module. Using [13, Theorem 2.5.1], G is also finitely presented since $R[[X]]$ is coherent. From Lemma 2.4, we have $\mathrm{Tor}_{R[[X]]}(M,R)=0$ since X is a nonzero divisor on M. In addition, $\mathrm{Tor}_{R[[X]]}(P,R)=0$ since P is a projective $R[[X]]$-module. Therefore, $\mathrm{Tor}_{R[[X]]}(G,R)=0$ (since $\mathrm{fd}_{R[[X]]}R\leq 1$). So, by Lemma 2.4, X is a nonzero divisor on G. On the other hand, if we tensor the short exact sequence above with $-\otimes_{R[[X]]}R$ we obtain a short exact sequence

$$0\longrightarrow G/XG\longrightarrow P/XP\longrightarrow M/XM\longrightarrow 0$$

(note that $M\otimes_{R[[X]]}R\cong M/XM$). Therefore, by the hypothesis condition of induction, $\mathrm{Gpd}_{R[[X]]}(G)\leq \mathrm{Gpd}_R(G/XG)\leq n-1$. Thus, $\mathrm{Gpd}_{R[[X]]}(M)\leq \mathrm{Gpd}_{R[[X]]}(G)+1\leq n$, as desired. □

Definition 2.6 ([18] and [12])**.** Let R be a ring and let M be an R-module.

(1) We say that M has FP-injective dimension at most n (for some $n\geq 0$), denoted by $\mathrm{FP\text{-}id}_R(M)\leq n$, if $\mathrm{Ext}_R^{n+1}(P,M)=0$ for every finitely presented R-module P.
(2) A ring R is said to be $n-FC$, if it is coherent and it has self-FP-injective at most at n (i.e., $\mathrm{FP\text{-}id}_R(R)\leq n$).

A ring is called FC ring if it is 0-FC.
Using [5, Theorems 6 and 7], we deduce the following Lemma.

Lemma 2.7. *Let R be a coherent ring and let $n \geq 0$ be an integer. The following are equivalent:*

(1) R is $n - FC$;
(2) $Gwdim\,(R) \leq n$;
(3) $Gpd_R(M) \leq n$ for every finitely presented R-module M.

Remark 2.8. (1) By Lemma 2.7, the Gorenstein weak dimension of a coherent ring R is also determined by the formula:

$$\mathrm{Gwdim}\,(R) = \ sup\{ \ \mathrm{Gpd}_R(M)|\ M \text{is a finitely presented } R\text{-module}\}.$$

(2) In Lemma 2.7, the case $n = 0$ (i.e., if R is FC) does not need the coherence condition (see [5, Theorem 6]).

Lemma 2.9. *Let R be a coherent ring and let X be an indeterminate over R. Then, $Gwdim\,(R[[X]]) \geq Gwdim\,(R) + 1$.*

Proof. By [13, Theorem 4.1.1(1)], $R \cong R[[X]]/XR[[X]]$ is coherent since it is a finitely presented R-module (from the short exact sequence
$0 \longrightarrow R[[X]] \xrightarrow{X} R[[X]] \longrightarrow R \longrightarrow 0$).
We may assume that $\mathrm{Gwdim}\,(R[[X]]) = n < \infty$. Using [13, Theorem 1.3.3] and [14, Proposition 2.27], we have $\mathrm{Gpd}_{R[[X]]}(R) = \mathrm{pd}_{R[[X]]}(R) = 1$. Thus, by Lemma 2.7, $\mathrm{Gwdim}\,(R[[X]]) = n \geq 1$ since R is a finitely presented $R[[X]]$-module.
Now, let M be a finitely presented R-module. Then, by [13, Theorem 2.1.8], M is a finitely presented $R[[X]]$-module (since $R \cong R[[X]]/XR[[X]]$). Thus, by [13, Theorem 1.3.5] and Lemma 2.7, $\mathrm{Ext}^n_R(M, R) = \mathrm{Ext}^{n+1}_{R[[X]]}(M, R[[[X]]) = 0$. Therefore, R is $(n-1) - FC$. Hence, by Lemma 2.7, $\mathrm{Gwdim}\,(R) \leq n - 1$. Therefore $\mathrm{Gwdim}\,(R) \leq \mathrm{Gwdim}\,(R[[X]]) - 1$, as desired. □

Now we are ready to present our main result of this paper.

Theorem 2.10. *Let R be a ring and let x be an indeterminate over R. If $R[[x]]$ is a coherent ring, then $Gwdim\,(R[[x]]) = Gwdim\,(R) + 1$.*

Proof. If $\mathrm{Gwdim}\,(R) = \infty$, then by Lemma 2.9, we have the desired equality. Otherwise we put $\mathrm{Gwdim}\,(R) = n$. By Lemma 2.9, it is enough to show that $\mathrm{Gwdim}\,(R[[X]]) \leq \mathrm{Gwdim}\,(R) + 1$. Let M be a finitely presented $R[[X]]$-module and consider a short exact sequence of $R[[X]]$-modules

$$(*) \ \ 0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0,$$

where P is a finitely generated projective $R[[X]]$-module. Then, by [13, Theorem 2.5.1], K is also finitely presented since $R[[X]]$ is coherent. Thus K/XK is also finitely presented R-module (by [13, Theorem 2.1.8]). On the other hand, from the short sequence $(*)$ we have $\mathrm{Tor}_{R[[X]]}(K, R) = \mathrm{Tor}^2_{R[[X]]}(M, R) = 0$ since $\mathrm{fd}_{R[[X]]}(R) \leq 1$. So, from Lemma 2.4, X is a nonzero divisor on K. Then, by Lemma 2.5 and Lemma 2.7, $\mathrm{Gpd}_{R[[X]]}(K) \leq \mathrm{Gpd}_R(K/XK) \leq n$. Then, $\mathrm{Gpd}_{R[[X]]}(M) \leq n+1$. Consequently, by Lemma 2.7, $\mathrm{Gwdim}\,(R[[X]]) \leq n+1 = \mathrm{Gwdim}\,(R) + 1$. □

Recall that a ring is called quasi-Frobenius, if it is Noetherian and self-injective (see [16]).

Proposition 2.11. *Let R be a quasi-Frobenius ring. Then,*

$$Gwdim\,(R[[X]]) = Ggldim\,(R[[X]]) = 1$$

Proof. Let R be a quasi-Frobenius ring. Then, from [4, Proposition 2.8 and Theorem 2.9], $\mathrm{Gwdim}\,(R) = \mathrm{Ggldim}\,(R) = 0$. Thus, from Theorem 2.10, $\mathrm{Gwdim}\,(R[[X]]) = 1$. On the other hand, R is Noetherian and so $R[[X]]$ is also Noetherian. Therefore, by [4, Theorem 2.9], $\mathrm{Ggldim}\,(R[[X]]) = 1$. $\square$

Recall that a ring R is called an arithmetical ring if every finitely generated ideal is locally principal. If $\mathrm{wdim}\,(R) \leq 1$, then R is an arithmetical ring (see for instance [2]). So we lead to ask the following question: If $\mathrm{Gwdim}\,(R) \leq 1$, then is R arithmetical ring?

The following result shows that the above question is false in general.

Theorem 2.12. *Let $(R, \mathfrak{m})$ be a local quasi-Frobenius ring which is not a field. Then the following statements hold:*

(1) *$Gwdim\,(R[[X]]) = 1$.*
(2) *$R[[X]]$ is not an arithmetical ring.*

Proof. (1) We have $\mathrm{Gwdim}\,(R[[X]]) = 1$ by Proposition 2.11.

2) We claim that $R[[X]]$ is not an arithmetical ring. Deny. Let a be a non-zero non-invertible element of R and let $I := aR[[X]] + XR[[X]]$. Then $I = PR[[X]]$ for some $P := \sum_i a_iX^i \in R[[X]]$ (where $a_i \in R$), since $R[[X]]$ is a local arithmetical ring.

Since $P \in I = aR[[X]] + XR[[X]]$, we have $P = aQ_1 + XQ_2$ for some $Q_1(:= \sum_i c_iX^i)$, $Q_2 \in R[[X]]$. Hence, $a_0 = ac_0$.

On the other hand, we have $a = PQ$ for some $Q = \displaystyle\sum_i b_iX^i \in R[[X]]$ (where $b_i \in R$) since $a \in I = PR[[X]]$. Hence, $a = a_0b_0$. We claim that $b_0 \in M$. If this is not the case, then b_0 is invertible in R and so Q is invertible in $R[[X]]$; hence, we may assume that $P = a$ (since $aR[[X]] = PQR[[X]] = PR[[X]] = I$). But, $X \in I = aR[[X]]$ implies that $X = a\sum_i d_iX^i$ for some $d_i \in R$. Hence, $1 = ad_1$ and so a is invertible in R, a contradiction. Therefore, $b_0 \in M$.

Therefore, $a = ab_0c_0$ since $a_0 = ac_0$ and $a = a_0b_0$ and so $a(1 - b_0c_0) = 0$. But $1 - b_0c_0$ is invertible in R since $b_0c_0 \in M$ (since $b_0 \in M$); hence $a = 0$, a contradiction. Hence, $R[[X]]$ is not an arithmetical ring, as desired. $\square$

In the rest of this paper, We compare the small finitistic Gorenstein projective dimension of the base ring R,

$$\mathrm{fGPD}\,(R) = \sup\left\{ \begin{array}{c|l} \mathrm{Gpd}_R(M) & M\ is\ an\ R-module\ with \\ & finite\ Gorenstein\ projective\ dimension \\ & and\ M\ admits\ a\ finite\ projective\ resolution \end{array} \right\}.$$

with the usual small finitistic projective dimension, $\mathrm{fPD}\,(R)$ (see [13]).

It is clear that if R is coherent we have

$$\mathrm{fGPD}\,(R) = sup\{\mathrm{Gpd}_R M \mid M \text{is a finitely presented and } \mathrm{Gpd}_R M < \infty\} \leq \mathrm{Gwdim}\,(R),$$

with equality if $\mathrm{Gwdim}\,(R)$ is finite (by Lemma 2.7).

In the proof of the next Theorem, we use the proofs of [14, Theorems 2.10 and 2.28].

Theorem 2.13. *For any coherent ring R there is an equality $fGPD\,(R) = fPD\,(R)$.*

Recall that a right co-proper projective resolution of an R-module M is an exact sequence $\mathbf{X} = 0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^1 \longrightarrow ...$ with X^i is projective for each $i \geq 0$ such that $\mathbf{Hom}_R(\mathbf{X}, P)$ is exact for every projective module P.

Lemma 2.14. *Let R be a ring. Then, every finitely generated G-projective R-module admits a right co-proper resolution of finitely generated free R-module.*

Proof. Let M be a finitely generated G-projective R-module. By [14, Proposition 2.4], there is an exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow L \longrightarrow M' \longrightarrow 0$$

where L is a free R-module and M' a G-projective R-module. We identify M to a submodule of L and we assume that L admits basis $\{x_k,\ k \in K\}$. Since M is finitely generated and each generator of M is a finite linear combination of finite subset of $\{x_k,\ k \in K\}$, we consider L_0 a finitely generated free direct summand of L which contains M and a free R-module L_1 such that $L = L_0 \oplus L_1$. Then, we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & 0 \longrightarrow & M & \hookrightarrow & L_0 & \longrightarrow & L_0/M & \longrightarrow 0 \\
 & & \| & & \downarrow & & & & \\
(1) & 0 \longrightarrow & M & \hookrightarrow & L & \longrightarrow & M' & \longrightarrow 0 \\
 & & & & \downarrow & & & & \\
 & & & & L_1 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

From [17, Exercise 2.7 page 29] the diagram above can be completed as

$$\begin{array}{cccccccc}
 & & & & 0 & & 0 & \\
 & & & & \downarrow & & \downarrow & \\
 & 0 \longrightarrow & M & \longrightarrow & L_0 & \longrightarrow & L_0/M & \longrightarrow 0 \\
 & & \| & & \downarrow & & \downarrow & \\
(2) & 0 \longrightarrow & M & \longrightarrow & L & \longrightarrow & M' & \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & \\
 & & & & L_1 & = & L_1 & \\
 & & & & \downarrow & & \downarrow & \\
 & & & & 0 & & 0 &
\end{array}$$

In the right vertical exact sequence L_1 is a projective module and so $M' \cong L_0/M \oplus L_1$. Then, from [14, Theorem 2.5], L_0/M is a G-projective R-module. In addition, by [14, Theorem 2.20], for every projective R-module F, the short sequence

$$0 \longrightarrow \mathrm{Hom}_R(L_0/M, F) \longrightarrow \mathrm{Hom}_R(L_0, F) \longrightarrow \mathrm{Hom}_R(M, F) \longrightarrow 0$$

is exact (since $\mathrm{Ext}(L_0/M, F) = 0$). On the other hand, it is clear that L_0/M is finitely generated. Then, by repeating this procedure, M admits a right co-proper resolution of finitely generated free R-modules. □

Proof of Theorem 2.13. Clearly $\mathrm{fPD}(R) \leq \mathrm{fGPD}(R)$ by [14, Proposition 2.27]. In first we claim that $\mathrm{fGPD}(R) \leq \mathrm{fPD}(R) + 1$. So, let M be a finitely presented module with $0 < \mathrm{Gpd}_R(M) = n < \infty$. We may pick an exact sequence, $0 \longrightarrow$

$K' \longrightarrow P_{n-1} \longrightarrow ... \longrightarrow P_0 \longrightarrow M \longrightarrow 0$, where $P_0, ..., P_{n-1}$ are finitely generated projective and K' is finitely generated G-projective (since R is coherent and by [14, Proposition 2.7]). On the other hand, by Lemma 2.14, there is an exact sequence $0 \longrightarrow K' \longrightarrow L^0 \longrightarrow ... \longrightarrow L^{n-1} \longrightarrow G \longrightarrow 0$ where $L^0, ..., L^{n-1}$ are finitely generated free module and G is finitely generated G-projective modules and such that $\mathrm{Hom}_R(-, Q)$ leaves this sequence exact, whenever Q is projective. Thus, by [14, Proposition 1.8] there exist homomorphisms, $L^i \to P_{n-1}$ for $i = 0, ..., n-1$, and $G \to M$, such that the following diagram is commutative.

$$\begin{array}{ccccccccccccc} 0 \longrightarrow & K' & \longrightarrow & L^0 & \longrightarrow & ... & \longrightarrow & L^{n-1} & \longrightarrow & G & \longrightarrow 0 \\ & \| & & \downarrow & & & & \downarrow & & \downarrow & \\ 0 \longrightarrow & K' & \longrightarrow & P_{n-1} & \longrightarrow & ... & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow 0 \end{array}$$

The diagram gives a chain map between complexes,

$$\begin{array}{ccccccccccc} 0 \longrightarrow & L^0 & \longrightarrow & ... & \longrightarrow & L^{n-1} & \longrightarrow & G & \longrightarrow 0 \\ & \downarrow & & & & \downarrow & & \downarrow & \\ 0 \longrightarrow & P_{n-1} & \longrightarrow & ... & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow 0 \end{array}$$

which induces an isomorphism in homology. Its mapping cone is exact and all the modules in it, except for $P_0 \oplus G$ (which is finitely generated Gorenstein projective), are finitely presented projective. Hence the kernel K of $\varphi : P_0 \oplus G \twoheadrightarrow M$ satisfies $\mathrm{pd}_R K \leq n-1$ (and then necessarily $\mathrm{pd}_R K = n-1$ by [14, Propositions 2.18 and 2.27]) and it is finitely presented since all P_i and L^i are finitely generated. Thus, we get $\mathrm{fGPD}(R) \leq \mathrm{fPD}(R) + 1$. Proving the inequality $\mathrm{fGPD}(R) \leq \mathrm{fPD}(R)$, we may therefore assume that $0 < \mathrm{fGPD}(R) = m < \infty$.

Pick a finitely presented module M with $\mathrm{Gpd}_R M = m$. We wish to find a finitely presented module Q with $\mathrm{pd}_R Q = m$. By the above proof there is an exact sequence

$$0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0$$

where G is a finitely generated Gorenstein projective module and K is a finitely presented module such that $\mathrm{pd}_R K = m-1$. Since G is a finitely generated Gorenstein projective module, there exists, by Lemma 2.14, a finitely generated free module L with $G \subseteq L$, and since also $K \subseteq G$, we can consider the quotient $Q = L/K$ (Q is finitely presented by [13, Theorem 2.5.1]). Note that $M \cong G/K$ is a submodule of Q, and that we get a short exact sequence

$$0 \longrightarrow M \longrightarrow Q \longrightarrow Q/M \longrightarrow 0.$$

If Q is Gorenstein projective, [14, Proposition 2.18] implies that $\mathrm{Gpd}_R(Q/M) = m+1$, since $\mathrm{Gpd}_R M = m$. But this contradict the fact that $m = \mathrm{fGPD}(R) < \infty$ since Q/M is finitely presented (by [13, Theorem 2.5.1] since Q and M are finitely presented and R is coherent). Hence Q is not Gorenstein projective, in particular, Q is not projective. Therefore the short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow Q \longrightarrow 0$ shows that $\mathrm{pd}_R Q = \mathrm{pd}_R K + 1 = m$.

□

Proposition 2.15. *let R be a coherent ring and let x be a nonzero divisor in R contained in the intersection of the maximal ideals of R. Then:*

(1) $fGPD(R) = fGPD(R/xR) + 1$
(2) If $Gwdim(R) < \infty$, then $Gwdim(R) = Gwdim(R/xR) + 1$

Proof. The first equality follows from Theorem 2.13 and [13, Corollary 3.1.4]. Now assume that $\mathrm{Gwdim}\,(R) = n$ is finite. We claim that $\mathrm{Gwdim}\,(R/xR)$ is finite. In first see, by [13, Theorem 4.1.1(1)], that R/xR is also coherent since R/xR is a finitely presented R-module (from the short exact sequence $0 \longrightarrow R \stackrel{x}{\longrightarrow} R \longrightarrow R/xR \longrightarrow 0$). Using [13, Theorem 1.3.3] and [14, Proposition 2.27], we have

$$\mathrm{Gpd}_R(R/xR) = \mathrm{pd}_R(R/xR) = 1$$

Then, from Lemma 2.7, $\mathrm{Gwdim}\,(R) = n \geq 1$ since R/xR is a finitely presented R-module. Now, let M be a finitely presented R/xR-module. Then, by [13, Theorem 2.1.8], M is a finitely presented R-module. Thus, by [13, Theorem 1.3.5] and Lemma 2.7, $\mathrm{Ext}^n_R(M, R/xR) = \mathrm{Ext}^{n+1}_R(M, R) = 0$. Therefore, R/xR is $(n-1) - FC$. Hence, by Lemma 2.7, $\mathrm{Gwdim}\,(R/xR) \leq n-1 < \infty$. So, by Theorem 2.13 and [13, Corollary 3.1.4], we have

$$\mathrm{Gwdim}\,(R) = \mathrm{fGPD}\,(R) = \mathrm{fGPD}\,(R/xR) + 1 = \mathrm{Gwdim}\,(R/xR) + 1.$$

Now the assertion holds. □

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